Geometry Problems from the IMO Shortlist

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1 Ideas to Try

Ideas to try on geometry problems:

1. *Angle Chasing*: Choose a set of angles that defines the diagram and find all possible angles in terms of them *e.g.* using cyclic quadrilaterals, similar triangles, common angle formulas.
2. *Length Chasing*: Find relationships between the lengths of sides *e.g.* using power of a point, similar triangles, Menelaus and Ceva, incircle and excircle side lengths, Pythagorean Theorem.
3. *Reduce the Problem*: Make some observations that reduce the problem to an easier problem or conjecture something plausible that implies the problem statement.
4. *Phantom Points*: To prove that a point P has a property, define a new point P' in a way that is easier to work with, then prove the property for P' and prove that $P = P'$.
5. *Combine Patterns*: Bring parts of the diagram that are related to each other together *e.g.* through parallel lines, intersecting circumcircles, reflections, constructing similar triangles.
6. *Spiral Similarity*: Look for or construct similar triangles of the form AOB and COD and use the angle and length relationships from the fact that AOC and BOD are also similar.
7. *Transformations*: Look for any transformations already present in the diagram and apply them to other parts of the diagram *e.g.* homothety, translation, reflection, spiral similarity.
8. *Constructing Points*: An introduced point P generally is useful if it has two "good" properties *i.e.* unites two conditions in the problem. Since a point P can always be selected to have a single property, introducing a point is only useful when it unites two conditions. However, most points introduced to solve problems are likely motivated by an approach listed above.
9. *Forming Conjectures*: Many difficult problems will require a lemma which may not be obvious from the problem statement or initial deductions. Two ways of forming conjectures are:
   (a) looking for patterns in precisely drawn diagrams and
   (b) thinking about what would be convenient and easy to work with if it were true

   It is important to keep both of these ideas in mind when looking for a key observation. Observations made only from the diagram may not be feasible to prove, useless to the problem or false. Conjectures that would be convenient may be obviously disproved by a diagram. It is important to conjecture something which seems clearly true based on one good (or several) diagrams and is both feasible to prove and useful in the problem.

10. *What is Difficult?*: A diagram will likely have parts that are difficult and parts that are easier to work with. It is often useful to identify what parts are difficult to work with and try to figure out possible ways to handle them *e.g.* redefining points using phantom points.

11. *Trigonometry*: Powerful in situations when an angle cannot be expressed simply in terms of other angles *e.g.* angles involving medians; often works best when you have in mind exactly what you want to prove *e.g.* a ratio condition.

12. *Algebraic Methods*: Complex numbers, vectors, coordinates and barycentric coordinates.

2 Examples

The examples below are intended to be representative of the types of problems that might appear on the IMO. The solutions given are outlines intended to emphasize motivation. We do not deal with configuration issues and special cases in the solutions presented.

Example 1. *(2004 G1) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.*

Solution. The initial difficulty with this problem is that defining R as the intersection of two unrelated angle bisectors does not give much information. We search for a better way to describe R. Since O is the center of the circle through $BMNC$, it follows that $OM = ON$ and the bisector of $\angle MON$ is the perpendicular bisector of MN. Now the bisector of $\angle BAC$ and perpendicular bisector of BC meet at the midpoint of arc $\widehat{BC}$. Therefore $AMRN$ is cyclic. If the circumcircles meet at P, angle chasing gives that B, P and C are collinear. □

The next example involves reducing the problem statement and parts of the diagram we need to consider as well as introducing phantom points.

Example 2. *(1995 G1) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.*

Solution. The diagram is cluttered and we try to reduce the parts of the diagrams we need to consider. The line AM is simply the perpendicular to CP at M and DN is simply the perpendicular to BP at N. We no longer have to think about A and D in defining these lines. Now we observe that since ZP is perpendicular to BC, these lines create cyclic quadrilaterals. It seems natural to introduce their intersections with ZP. Let the perpendiculars at M and N to CP and BP intersect ZP at Q and R. We now have that $ZXMC$ and $ZYNB$ are cyclic. Power of a point yields that $PQ \cdot PZ = PM \cdot PC = PX \cdot PY = PN \cdot PB = PR \cdot PZ$. Therefore $Q = R$ and we are done. □

One important note with eliminating parts of the daigram is that you may lose information or those parts may motivate the solution. It is important to consider the problem both with and without unnecessary parts of the diagram. The next problem exemplifies the method of completing a transformation already present in the diagram. In an trapezoid, there is an internal and an external homothety mapping the two parallel sides to one another. In this problem, we "complete" the transformation by filling in the missing point-transform pairs.

Example 3. *(2006 G2) Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $\frac{AK}{KB} = \frac{DL}{LC}$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B and C are concyclic.*

Solution. Since $ABCD$ is a trapezoid, there is a homothety sending AB to CD as well as one sending AB to DC. We note that the homothety sending AB to DC also sends K to L. Now we complete this homothety in the diagram. Let DA and CB intersect at T and let the homothety with center T bring P to P'. We have that K, P, Q, L and P' are collinear and $PB \| P'C$. Since $\angle DQC + \angle APB = \angle DQC + \angle DP'C = 180°$, we have $DQCP'$ is cyclic. Therefore $\angle QPB = \angle QP'C = \angle QDC = 180° - \angle DQC - \angle QCD = \angle QCB$. The conclusion follows. □

The next problem really illustrates the power of looking for similar triangles and stopping to think about what is already in the diagram before trying to introduce new points.

Example 4. *(2005 G3) Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A-excenters of the triangles ABX and ADY. Show that $\angle KCL$ is independent of the line g.*

Solution. Angle chasing gives that $\angle ALD = \angle KAB = \angle BAX/2$ and $\angle DAL = \angle BKA = \angle ADY/2$. Therefore triangles ADL and KBA are similar which implies that $AB/BK = DL/AD$ and therefore $DL/CD = BC/BK$. Since $\angle CDL = \angle CBK = 90° - \angle ADC/2$, it follows that triangles CDL and KBC are siimilar. Now it follows that $\angle KCL = 360° - \angle BCD - \angle DCL - \angle BCK = 180° + \angle CDL - \angle BCD = 180° - \angle BCD/2$ which is independent of g. □

The next example illustrates the power of redefining a point that is difficult to work with. Here, working with the problem defined from an easier point of view reduces it to angle chasing.

Example 5. *(2002 G3) The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.*

Solution. We first make several preliminary observations. Since EF is the perpendicular bisector of OA, we have that $AE = OE = OA$ and therefore AOE is equilateral. Similarly, we have that AOF is equilateral which implies that $\angle EOF = 120°$ and $\angle ECF = 60°$. These results also imply that A is the midpoint of arc $\widehat{EF}$ and CA bisects $\angle ECF$. After these preliminary observations, it becomes difficult to work with the point I as defined. The key here is to redefine I to be easier to work with. We now define I' to be the incenter of CEF with the goal of showing that $\angle DAO = \angle AOI'$ since this would imply that $OI' \| AD$ and therefore $I = I'$. At this point, the task becomes far more feasible than before and reduces to angle chasing. First we note that $\angle EOF = 120°$ and $\angle EI'F = 90° + \angle ECF/2 = 120°$ which implies that $EI'OF$ is cyclic. Now we carry out our angle chasing methodically, attempting to eliminate points from consideration as we go. Note that $\angle DAO = 90° - \angle AOD/2 = 90° - \angle ACB/2 = 45° + \angle ABC/2 = 45° + \angle AFC/2$, which is enough to eliminate D and B. Now note that $\angle AOI' = \angle AOE + \angle EOI' = 60° + \angle EFI = 60° + \angle EFC/2$. Since $\angle AFC - \angle EFC = 30°$, we have that $\angle DAO = \angle AOI'$, as desired. □

The next example has multiple elements that are difficult to work with. Here, we follow cues presented in the diagram and obtain useful constructions (introduced points uniting more than one condition) and reduce the problem to feasible ratio calculations.

Example 6. *(1996 G3) Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\angle FHP = \angle BAC$.*

Solution. If the problem statement is true, then $\angle CHP = 180° - \angle BAC$. Based on this angle relationship, intersecting HP with AB creates a cyclic quadrilateral. We reformulate the problem by defining P as the point on AC satisfying $\angle FHP = \angle BAC$ introduce this intersection point and call it D. Our goal is now to show $\angle PFO = 90°$ and the two definitions are therefore equivalent. Since $CHAD$ is cyclic, we have that $\angle CDA = 180° - \angle CHA = \angle CBA$. Since the line OF is difficult to deal with and angles around it have no simple formula, we try to reduce the problem to a condition relating something more directly related to P than OF. We have now that DCB is isosceles and F is the midpoint of BD. If M is the midpoint of AB, then we now note that there is a homothety sending MF to AD with center B and ratio 2. Let E be the image of O under this homothety. Note that $AE = 2OM = CH$. It now suffices to show that $\angle EDA = 90° - \angle PFH$. We now try to reduce this angle condition to length conditions which will be easier to deal with since many angles in the diagram cannot be expressed simply. If G is the intersection of FP with the line through C perpendicular to CH. Since $\angle GCF = \angle EAD = 90°$, it suffices to show that GCF and EAD are similar, which is equivalent to showing that

$$\frac{CH}{AD} = \frac{EA}{AD} = \frac{GC}{CF} = \frac{CP}{PA} \cdot \frac{AF}{CF}$$

Now we resort to a ratio identity for cyclic quadrilaterals. The ratio CP/PA is the ratio of the areas of triangles DCH and DAH. Therefore since $CHAD$ is cyclic, we have that

$$\frac{CP}{PA} = \frac{\sin\angle DCH \cdot CD \cdot CH}{\sin\angle DAH \cdot AD \cdot AH} = \frac{CB \cdot CH}{AD \cdot AH}$$

Therefore the desired result reduces to proving that $AH/AF = BC/CF$ which follows from the fact that AHF and CBF are similar. This completes the proof. □

The next example demonstrates the effectiveness of persisting with a particular approach before moving on and introducing new points into the diagram.

Example 7. *(2008 G4) In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A anf F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.*

Solution. This problem is straightforward with power of a point and does not require introducing any new points other than the orthocenter H of ABC and foot of the perpendicular from A to BC, which are already implicitly present. Relating our goal to angles already in the diagram reduces the problem to showing that $\angle QFB = \angle PEC$. By power of a point $BQ^2 = BP^2 = BF \cdot BA$ and triangles QFB and AQB are similar. Therefore it suffices to show that $\angle PEC = \angle AQC$ which is equivalent to $AQPE$ being cyclic. By power of a point we now have

$$CP \cdot CQ = BC^2 - BP^2 = BC^2 - BF \cdot BA = BC^2 - BD \cdot BC = CD \cdot CB = CE \cdot CA$$

Therefore $AQPE$ is cyclic and we are done. □

This solution hides the experimenting involved with power of a point needed to come up with it. Although it is tempting to try introducing new points, here just persisting with what is already present solves the problem. The key idea in the next example is to find an easier condition to work with and to combine related ideas.

Example 8. *(2006 G4) Let ABC be a triangle such that $\widehat{ACB} < \widehat{BAC} < \frac{\pi}{2}$. Let D be a point of $[AC]$ such that $BD = BA$. The incircle of ABC touches $[AB]$ at K and $[AC]$ at L. Let J be the center of the incircle of BCD. Prove that (KL) intersects $[AJ]$ at its middle.*

Solution. Angle chasing gives that $\angle ALK = 90^\circ - \angle A/2$ and $\angle CDJ = 90^\circ - \angle A/2$. It makes sense to try to relate these two equal angles in the diagram by trying to move one into a position so that it relates to the other. Furthermore, working on the segment AJ seems difficult as we do not know angles or lengths related to this line. Instead, we try to work on AC, where we can make use of incircle tangent length formulas. We do this by reducing the problem using non-perpendicular projections in the direction of KL onto AC. We find that this reduces the problem to a seemingly feasible alternative and also relates the equal angles originally found. Specifically, let P be the intersection of the line perpendicular to KL through J with AC. It now suffices to show that L is the midpoint of AP. Since $\angle PDJ = \angle ALK = \angle DPJ$, we have that PDJ is isosceles and if M is the midpoint of DP, then M is also the foot of the perpendicular from J onto AC. Applying incircle tangent length formulas gives that $AL = \frac{1}{2}(AB + AC - BC)$ and $AP = AD + 2AM = AD + (BD + DC - BC) = AB + AC - BC$. This implies that L is the midpoint of AP and the desired result follows. □

The next example has a key lemma not at all obvious from the problem statement. We try to motivate how to find this observation.

Example 9. *(2005 G5) Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.*

Solution. It is a known fact that the line HM passes through P, the point diametrically opposite to A on the circumcircle Γ of ABC. Based on this, it would be convenient if HM passed through the second intersection Q of Γ and the circumcircle of ADE. If this were true, then AQ and the line $\overline{PMHQ}$ would be perpendicular since AP is a diameter of the circumcircle of ABC. At this point, it is not a bad idea to draw one or two precise diagrams and see if our claim is supported. We find that it is and decide to focus on this claim. Proving the claim directly does not seem easy since it is hard to work with the second intersection point while actually using the fact that it lies on both circles. We look for a conjecture easier to prove that arises from our claim. If the claim is true, then $\overline{PMHQ}$ must also pass through the point R diametrically opposite to A on the circumcircle of ADE. Proving this seems more feasible, since it does not involve the second intersection and we work with it first. Treating this new claim as its own subproblem yields the following solution.

Let U and V be the feet of the perpendiculars from B and C to AC and AB. Angle chasing yields that the line $\overline{DHE}$ is the internal bisector of the angle formed by lines BU and CV. It also holds that triangles UHC and VHB are similar. Therefore $UD/DB = VE/EC = t$. If the perpendicular to AB at D intersects HP at R_1, then since $UHPB$ is a trapezoid it follows that

$HR_1/R_1P = t$. Similarly if the perpendicular to AC at E intersects HP at R_2, then $HR_2/R_2P = t$ and $R_1 = R_2 = R$. This proves the claim.

Now to complete the solution, take the projection Q' of A onto line $\overline{PMHR}$. Since AR and AP are diameters of the circumcircle of ADE and Γ, it follows that Q' lies on both circles and thus $Q' = Q$. Now it follows that the line $\overline{PMHR}$ is perpendicular to AQ, as desired. □

3 Problems

In this section, a variety of IMO Shortlist problems are included. They are intended to be sorted roughly in increasing order of difficulty.

A1. (2003 G1) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

A2. (2002 G1) Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

A3. (1998 G1) A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

A4. (2001 G1) Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

A5. (2000 G1) Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

A6. (2003 G2) Given three fixed pairwisely distinct points A, B, C lying on one straight line in this order. Let G be a circle passing through A and C whose center does not lie on the line AC. The tangents to G at A and C intersect each other at a point P. The segment PB meets the circle G at Q. Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G.

A7. (2008 G1) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

A8. (2001 G2) Consider an acute-angled triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$.

A9. (2005 G2) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

B1. (2006 G3) Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE \quad , \quad \angle ABC = \angle ACD = \angle ADE$$

Let P be the point of intersection of the lines BD and CE. Prove that the line AP passes through the midpoint of the side CD.

B2. (2009 G2) Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

B3. (2012 G3) In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

B4. (2007 G3) The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

B5. (2000 G3) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

B6. (2009 G4) Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

B7. (1998 G6) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

B8. (2009 G3) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

B9. (2003 G5) Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

B10. (2005 G4) Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

B11. (1995 G8) Suppose that $ABCD$ is a cyclic quadrilateral. Let $E = AC \cap BD$ and $F = AB \cap CD$. Denote by H_1 and H_2 the orthocenters of triangles EAD and EBC, respectively. Prove that the points F, H_1, H_2 are collinear.

B12. (2007 G4) Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

B13. (2011 G4) Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

B14. (2010 G5) Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

C1. (1998 G5) Let ABC be a triangle, H its orthocenter, O its circumcenter, and R its circumradius. Let D be the reflection of the point A across the line BC, let E be the reflection of the point B across the line CA, and let F be the reflection of the point C across the line AB. Prove that the points D, E and F are collinear if and only if $OH = 2R$.

C2. (1998 G8) Let ABC be a triangle such that $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to the circumcircle ω of triangle ABC meets the line BC at D. Let E be the reflection of A in the line BC, let X be the foot of the perpendicular from A to BE, and let Y be the midpoint of the segment AX. Let the line BY intersect the circle ω again at Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.

C3. (1999 G6) Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

C4. (2005 G6) Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

C5. (2004 G7) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

C6. (2009 G6) Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

C7. (1996 G5) Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

C8. (2011 G3) Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA and AB. Prove that the midpoint of the segment EF lies on the line through the two intersections of ω_E and ω_F.

C9. (2008 G7) Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to AD and CD. Prove that the common external tangents to k_1 and k_2 intersect on k.

C10. (2006 G9) Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively $(A_2 \neq A, B_2 \neq B, C_2 \neq C)$. Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

C11. (2012 G6) Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD+BF = CA$ and $CD+CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

C12. (2007 G8) Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

C13. (2009 G8) Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

C14. (2004 G8) Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

4 Useful Geometry Facts

Cyclic Quadrilaterals

1. A convex quadrilateral $ABCD$ is cyclic if and only if either:
   (a) $\angle ADB = \angle ACB$
   (b) $\angle DAB + \angle BCD = 180°$
2. The above two conditions can be restated as a single condition in terms of directed angles: Four points A, B, C and D are concyclic if and only if $\measuredangle ABC = \measuredangle ADC$.
3. (Power of a Point) Let $ABCD$ be a convex quadrilateral such that AB and CD intersect at P and diagonals AC and BD intersect at Q. $ABCD$ is cyclic if and only if either:
   (a) $AQ \cdot QC = BQ \cdot QD$ or equivalently QAD and QBC are similar
   (b) $PA \cdot PB = PC \cdot PD$ or equivalently PAD and PCB are similar
4. Given a triangle ABC, the intersections of the internal and external bisectors of angle $\angle BAC$ with the perpendicular bisector of BC both lie on the circumcircle of ABC.
5. (Ptolemy's Theorem) A quadrilateral $ABCD$ is cyclic if and only if
$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$
6. Let $ABCD$ be a cyclic quadrilateral such that AB and CD intersect at P and diagonals AC and BD intersect at Q. Then:
$$\frac{BQ}{QD} = \frac{AB \cdot BC}{AD \cdot DC} \quad \text{and} \quad \frac{PB}{PA} = \frac{BC \cdot BD}{AC \cdot AD}$$
7. (Polars) Let $ABCD$ be a cyclic quadrilateral inscribed in circle Γ such that AB and CD intersect at P and diagonals AC and BD intersect at Q. If the tangents drawn from P to Γ touch Γ at R and S, then R, Q and S are collinear.

Circles

1. (Power of a Point) Given a circle Γ with center O and a point P then for any line ℓ through P that intersects Γ at A and B, the value $PA \cdot PB$ is constant as ℓ varies and is equal to the power of the point P with respect to Γ.
   (a) The power of P is equal to $r^2 - PO^2$ if P is inside Γ and $PO^2 - r^2$ otherwise.
   (b) If PA is tangent to Γ, then the power of P is equal to PA^2.
2. (Radical Axis) Given two circles Γ_1 and Γ_2, the set of all points P with equal powers with respect to Γ_1 and Γ_2 is a line which is the radical axis of the two circles.
   (a) The radical axis is perpendicular to the line through the centers of Γ_1 and Γ_2.
   (b) If Γ_1 and Γ_2 intersect at A and B, then the radical axis passes through A and B.
   (c) If AB is a common tangent with A on Γ_1 and B on Γ_2, then the radical axis passes through the midpoint of AB.
3. (Radical Center) Given three circles Γ_1, Γ_2 and Γ_3, the three radical axes between pairs of the three circles meet at a common point P which is the radical center of the circles.
4. A point P is a circle of radius zero and the radical axis of P and a circle Γ is the line through the midpoints of PA and PB where A and B are points on Γ such that PA and PB are tangent to Γ.
5. (Monge's Theorem) Given three circles Γ_1, Γ_2 and Γ_3. If P, Q and R are the external centers of homothety between pairs of the three circles, then P, Q and R are collinear. If P and Q are internal centers of homothety, then P, Q and R are also collinear.
6. Two circles Γ_1 and Γ_2 intersect at R and have centers O_1 and O_2. If P and Q are the internal and external centers of homothety between the two circles, then $\angle PRQ = 90^\circ$. The lines RP and RQ are the internal and external bisectors of $\angle O_1RO_2$.

Triangle Geometry

1. (Angle Bisector Theorem) Let ABC be a given triangle and let P and Q be the intersections of the internal and external bisectors of angle $\angle ABC$ with line AC. Then
$$\frac{AB}{BC} = \frac{AP}{PC} = \frac{AQ}{QC}$$
2. Angles around the centers of a triangle ABC:
   (a) If I is the incenter of ABC then $\angle BIC = 90^\circ + \frac{a}{2}$, $\angle IBC = \frac{b}{2}$ and $\angle ICB = \frac{c}{2}$.
   (b) If H is the orthocenter of ABC then $\angle BHC = 180^\circ - a$, $\angle HBC = 90^\circ - c$ and $\angle HCB = 90^\circ - b$.
   (c) If O is the circumcenter of ABC then $\angle BOC = 2a$ and $\angle OBC = \angle OCB = 90^\circ - a$.
   (d) If I_a is the A-excenter of ABC then $\angle AI_aB = \frac{c}{2}$, $\angle AI_aC = \frac{b}{2}$ and $\angle BI_aC = 90^\circ - \frac{a}{2}$.

3. Pedal triangles of the centers of a triangle ABC:
   (a) If DEF is the triangle formed by projecting the incenter I onto sides BC, AC and AB, then I is the circumcenter of DEF and $\angle EDF = 90° - \frac{a}{2}$.
   (b) If DEF is the triangle formed by projecting the orthocenter H onto sides BC, AC and AB, then H is the incenter of DEF and $\angle EDF = 180° - 2a$.
   (c) The medial triangle of ABC is the pedal triangle of the circumcenter O of ABC and O is its orthocenter.
4. Alternate methods of defining the orthocenter and circumcenter:
   (a) O is the circumcenter of ABC if and only if $\measuredangle AOB = 2\measuredangle ACB$ and $OA = OB$.
   (b) H is the orthocenter of ABC if and only if H lies on the altitude from A and satisfies that $\measuredangle BHC = 180° - \measuredangle BAC$.
5. Facts related to the orthocenter H of a triangle ABC with circumcircle Γ:
   (a) If O is the circumcenter of ABC, then $\angle BAH = \angle CAO$.
   (b) If D is the point diametrically opposite to A on Γ and M is the midpoint of BC, then M is also the midpoint of HD.
   (c) If AH, BH and CH intersect Γ again at D, E and F, then there is a homothety centered at H sending the pedal triangle of H to DEF with ratio 2.
   (d) If D and E are the intersections of AH with BC and Γ, respectively, then D is the midpoint of HE.
   (e) H lies on the three circles formed by reflecting Γ about AB, BC and AC.
   (f) If M is the midpoint of BC then $AH = 2 \cdot OM$.
   (g) If BH and CH intersect AC and AB at D and E, and M is the midpoint of BC, then M is the center of the circle through B, D, E and C, and MD and ME are tangent to the circumcircle of ADE.
6. Facts related to the incenter I and excenters I_a, I_b, I_c of ABC with circumcircle Γ:
   (a) If the incircle of ABC is tangent to AB and AC at points D and E and s is the semiperimeter of ABC then
   $$AD = AE = \frac{AB + AC - BC}{2} = s - BC$$
   (b) If AI intersects Γ at D then $DB = DI = DC$, D is the midpoint of II_a, and II_a is a diameter of the circle with center D which passes through B and C.
   (c) If AI, BI and CI intersect Γ at D, E and F, then $I_aI_bI_c$, DEF and the pedal triangle of I are similar and have parallel sides.
   (d) I is the orthocenter of $I_aI_bI_c$ and Γ is the nine-point circle of $I_aI_bI_c$.
   (e) If BI and CI intersect Γ again at D and E, then I is the reflection of A in line DE and if M is the intersection of the external bisector of $\angle BAC$ with Γ, then $DMEI$ is a parallelogram.

(f) If the incircle and A-excircle of ABC are tangent to BC at D and E, $BD = CE$.

(g) If the A-excircle of ABC is tangent to AB, AC and BC at D, E and F then $AB+BF = AC + CF = AD = AE = s$ where s is the semi-perimeter of ABC.

(h) If M is the midpoint of arc BAC of Γ, then M is the midpoint of I_bI_c and the center of the circle through I_b, I_c, B and C.

7. (Nine-Point Circle) Given a triangle ABC, let Γ denote the circle passing through the midpoints of the sides of ABC. If H is the orthocenter of ABC, then Γ passes through the midpoints of AH, BH and CH and the projections of H onto the sides of ABC.

8. (Feuerbach's Theorem) The nine-point circle is tangent to the incircle and excircles.

9. (Euler Line) If O, H and G are the circumcenter, orthocenter and centroid of a triangle ABC, then G lies on segment OH with $HG = 2 \cdot OG$.

10. (Symmedian) Given a triangle ABC such that M is the midpoint of BC, the symmedian from A is the line that is the reflection of AM in the bisector of angle $\angle BAC$.

    (a) If the tangents to the circumcircle Γ of ABC at B and C intersect at N, then N lies on the symmedian from A and $\angle BAM = \angle CAN$.

    (b) If the symmedian from A intersects Γ at D, then $AB/BD = AC/CD$.

11. If the median from A in a triangle ABC intersects the circumcircle Γ of ABC at D, then $AB \cdot BD = AC \cdot CD$.

12. (Euler's Formula) Let O, I and I_a be the circumcenter, incenter and A-excenter of a triangle ABC with circumradius R, inradius r and A-exradius r_a. Then:

    (a) $OI = \sqrt{R(R-2r)}$.

    (b) $OI_a = \sqrt{R(R+2r_a)}$.

13. (Poncelet's Porism) Let Γ and ω be two circles with centers O and I and radii R and r, respectively, such that $OI = \sqrt{R(R-2r)}$. Let A, B and C be any three points on Γ such that lines AB and AC are tangent to ω. Then line BC is also tangent to ω.

14. (Apollonius Circle) Let ABC be a given triangle and let P be a point such that $AB/BC = AP/PC$. If the internal and external bisectors of angle $\angle ABC$ meet line AC at Q and R, then P lies on the circle with diameter QR.

15. Let ABC be a given triangle with incircle ω and A-excircle ω_a. If ω and ω_a are tangent to BC at M and N, then AN passes through the point diametrically opposite to M on ω and AM passes through the point diametrically opposite to N on ω_a.

16. Let ABC be a triangle with incircle ω which is tangent to BC, AC and AB at D, E and F. Let M be the midpoint of BC. The perpendicular to BC at D, the median AM and the line EF are concurrent.

17. Let ABC be a triangle with incenter I and incircle ω which is tangent to BC, AC and AB at D, E and F. The angle bisector CI intersects FE at a point T on the line adjoining the midpoints of AB and BC. It also holds that $BFTID$ is cyclic and $\angle BTC = 90°$.

Collinearity and Concurrency

1. (Ceva's Theorem) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an even number are on the extensions of the sides (zero or two). Then AD, BE and CF are concurrent if and only if
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$
2. (Menelaus' Theorem) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an odd number are on the extensions of the sides (one or three). Then D, E and F are collinear if and only if
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$
3. (Trig Ceva) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an even number are on the extensions of the sides (zero or two). Then AD, BE and CF are concurrent if and only if
$$\frac{\sin(\angle ABE)}{\sin(\angle CBE)} \cdot \frac{\sin(\angle BCF)}{\sin(\angle ACF)} \cdot \frac{\sin(\angle CAD)}{\sin(\angle BAD)} = 1$$
4. (Casey's Theorem) If A_1, B_1 and C_1 are points on the sides BC, AC and AB of a triangle ABC, then the perpendiculars to their respective sides at these three points are concurrent if and only if $BA_1^2 - CA_1^2 + CB_1^2 - AB_1^2 + AC_1^2 - BC_1^2 = 0$.
5. (Pascal's Theorem) If A, B, C, D, E, F are points on a circle then the intersections of the pairs of lines AB and DE, BC and EF, CD and FA lie on a line.
6. (Pappus' Theorem) If A, C and E lie on one line ℓ_1 and B, D and F lie on a line ℓ_2, then the intersections of the pairs of lines AB and DE, BC and EF, CD and FA lie on a line.
7. (Brianchon's Theorem) If $ABCDEF$ is a hexagon with an inscribed circle then AD, BE and CF are concurrent.
8. (Desargues Theorem) Let ABC and XYZ be triangles. Let D, E, F be the intersections of the pairs of lines AB and XY, BC and YZ, AC and XZ. Then D, E and F are collinear if and only if AX, BY and CZ are concurrent.
9. Pascal's theorem is true when points are not necessarily distinct and many of its applications concern tangent lines when some of the six points are equal.

Trigonometry

1. (Sine Law) Given a triangle ABC with circumradius R
$$\frac{BC}{\sin \angle A} = \frac{AC}{\sin \angle B} = \frac{AB}{\sin \angle C} = 2R$$

2. (Cosine Law) Given a triangle ABC

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos \angle A$$

3. (Pythagorean Theorem) If ABC is a triangle, then $\angle ABC = 90°$ if and only if

$$AB^2 + BC^2 = AC^2$$

4. Given a triangle ABC and a point D on line BC, then

$$\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{BD \cdot AC}{CD \cdot AB}$$

5. (Stewart's Theorem) Let a, b, c be the side lengths of a triangle ABC and let d be the length of a cevian from A to BC that divides BC into segments of lengths m and n with m closer to B. Then

$$b^2 m + c^2 n = a(d^2 + mn)$$

Miscellaneous Synthetic Facts

1. (Spiral Similarity) Let OAB and OCD be directly similar triangles. Then OAC and OBD are also directly similar triangles.

2. The unique center of spiral similarity sending AB to CD is the second intersection of the circumcircles of QAB and QCD where AC and BD intersect at Q.

3. Lines AB and CD are perpendicular if and only if $AC^2 - AD^2 = BC^2 - BD^2$.

4. (Apollonius Circle) Given two points A and B and a fixed $r > 0$, then the locus of points Q such that $AQ/BQ = r$ is a circle Γ with center at the midpoint of Q_1Q_2 where Q_1 and Q_2 are the two points on line AB satisfying $AQ_i/BQ_i = r$ for $i = 1, 2$.

5. Let $ABCD$ be a convex quadrilateral. The four interior angle bisectors of $ABCD$ are concurrent and there exists a circle Γ tangent to the four sides of $ABCD$ if and only if $AB + CD = AD + BC$.

6. (Simson Line) Let M, N and P be the projections of a point Q onto the sides of a triangle ABC. Then Q lies on the circumcircle of ABC if and only if M, N and P are collinear. If Q lies on the circumcircle of ABC, then the reflections of Q in the sides of ABC are collinear and pass through the orthocenter of the triangle.

7. (Broken Chord Theorem) Let E is the midpoint of major arc $\widehat{ABC}$ of the circumcircle of a triangle ABC where $AB < BC$. If D is the projection of E onto BC, then $AB + BD = DC$.

8. (Butterfly Theorem) Let M be the midpoint of a chord XY of a circle Γ. The chords AB and CD pass through M. If AD and BC intersect chord XY at P and Q, then M is also the midpoint of PQ.

9. (Miquel Point) Let D, E and F be points on sides BC, AC and AB of a triangle ABC. Then the circumcircles of AEF, BDF and CDE are concurrent.

10. (Isogonal Conjugates) Let ABC be a triangle and P be a point. If the reflection of BP in the angle bisector of $\angle ABC$ and the reflection of CP in the angle bisector $\angle ACB$ intersect at Q, then Q lies on the reflection of CP in the angle bisector of $\angle ACB$.

11. (Casey's Theorem) Let O_1, O_2, O_3, O_4 be four circles tangent to a circle O. Let t_{ij} be the length of the external common tangent between O_iO_j if O_i and O_j are tangent to O from the same side and the length of the internal common tangent otherwise. Then

$$t_{12} \cdot t_{34} + t_{41} \cdot t_{23} = t_{13} \cdot t_{24}$$

The converse is also true: if the above equality holds then O_1, O_2, O_3, O_4 are tangent to O.

12. (Transversal Theorem) If A, B and C are collinear and A', B' and C' are points on AP, BP and CP, then A', B' and C' are collinear if and only if

$$BC \cdot \frac{AP}{A'P} + CA \cdot \frac{BP}{B'P} + AB \cdot \frac{CP}{C'P} = 0$$

where all lengths are directed.

13. (Mixtilinear Incircles) Let ABC be a triangle with circumcircle Γ and let ω be a circle tangent internally to Γ and to AB anc AC at X and Y. Then the incenter of ABC is the midpoint of segment XY.

14. (Curvilinear Incircles) Let ABC be a triangle with circumcircle Γ and let D be a point on segment BC. Let ω be a circle tangent to Γ, DA and DC. If ω is tangent to DA and DC at F and E, then the incenter of ABC lies on FE.